Trace Formulas

Volume 1a
Classical Trace Formulas for the Modular Group

Steven Creech

Draft edition

Acknowledgments

This material is based upon work supported by the National Science Foundation under Award No. DMS-2602321.

Any opinions, findings and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the National Science Foundation.

Contents

Acknowledgments i

I Foundations 1

0 What Is a Trace Formula? 2
- 0.1 Finite matrices and two descriptions of a trace . . . 2
- 0.2 Finite-rank kernels and diagonal integration . . . 5
- 0.3 Ordinary traces, relative pairings, and weights . . . 9

1 The Upper Half-Plane and the Modular Group 14
- 1.1 Möbius transformations and the role of the center . . . 14
- 1.2 Hyperbolic metric, measure, distance, and geodesics . . . 14
- 1.3 Reduction to the standard fundamental domain . . . 14
- 1.4 The cusp, elliptic points, and stabilizers . . . 14
- 1.5 Quotient integration and the area calculation . . . 14

2 Fourier Analysis, Mellin Transforms, and Unfolding 15
- 2.1 Fourier series and coefficient extraction . . . 15
- 2.2 Fourier transforms and Poisson summation . . . 15
- 2.3 Gamma integrals and Mellin inversion . . . 15
- 2.4 A general unfolding lemma . . . 15
- 2.5 Convergence, domination, and truncation . . . 15

II Holomorphic forms and trace formulas 16

3 Holomorphic Modular Forms 17
- 3.1 Automorphy factors and slash actions . . . 17
- 3.2 The q-variable and holomorphy at infinity . . . 17
- 3.3 Modular forms, cusp forms, and Fourier expansions . . . 17
- 3.4 Growth and cusp decay . . . 17
- 3.5 The Petersson inner product . . . 17

4 Explicit Forms, Zeros, and Dimensions 18
- 4.1 Holomorphic Eisenstein series and their coefficients . . . 18
- 4.2 E_4, E_6, and the discriminant . . . 18

4.3 The valence formula, including elliptic points . . . 18
4.4 The ring of level-one modular forms . . . 18
4.5 Dimension formulas and explicit small-weight bases . . . 18

5 Hecke Operators and Their Two Normalizations 19
5.1 Integral matrices, determinant correspondences, and coset representatives . . . 19
5.2 The classical operators and their Fourier action . . . 19
5.3 Hecke multiplication and prime-power recurrences . . . 19
5.4 Adjointness and simultaneous eigenbases . . . 19
5.5 Classical versus normalized operators; eigenvalue conventions . . . 19

6 Poincaré Series, Kloosterman Sums, and Bessel Functions 20
6.1 Automorphization of a seed and Poincaré series . . . 20
6.2 Absolute and local uniform convergence . . . 20
6.3 Fourier coefficients as Petersson pairings . . . 20
6.4 Coset decomposition and Kloosterman sums . . . 20
6.5 The Bessel integral and the full Fourier expansion . . . 20

7 The Petersson Trace Formula 21
7.1 The definitive orthonormal-basis statement . . . 21
7.2 The Poincaré-series proof . . . 21
7.3 Hecke-normalized and harmonic-weight versions . . . 21
7.4 Convergence, small weights, and vanishing spaces . . . 21
7.5 Orthogonality, second moments, and consistency checks . . . 21

8 Holomorphic Kernels and Linear Functionals 22
8.1 Finite-dimensional reproducing kernels . . . 22
8.2 The holomorphic kernel on the upper half-plane . . . 22
8.3 Automorphization and the cuspidal kernel . . . 22
8.4 Kernels of Hecke operators . . . 22
8.5 Linear functionals, Fourier extraction, and relative pairings . . . 22

9 Quadratic Forms, Conjugacy Classes, and Class Numbers 23
9.1 Binary quadratic forms and discriminants . . . 23
9.2 Reduction and equivalence . . . 23
9.3 Automorphisms and weighted class numbers . . . 23
9.4 Matrices of determinant n and discriminant t^2-4n . . . 23
9.5 Scalar, elliptic, parabolic, and hyperbolic cases . . . 23

10 The Eichler–Selberg Trace Formula 24
10.1 What the Hecke trace measures . . . 24
10.2 The holomorphic-kernel calculation . . . 24
10.3 Geometric contributions and correction terms . . . 24
10.4 Dimension and one-dimensional-space checks . . . 24
10.5 A complementary period-polynomial proof . . . 24

III Maass forms and spectral theory **25**

11 The Hyperbolic Laplacian **26**
11.1 Elementary spectral models . . . 26
11.2 The hyperbolic Laplacian and invariance . . . 26
11.3 Hilbert spaces and unbounded operators . . . 26
11.4 Green identities, domains, and self-adjointness . . . 26
11.5 Regularity, compactness, and the cusp . . . 26

12 Maass Forms and Their Fourier Expansions **27**
12.1 Automorphic eigenfunctions and cusp conditions . . . 27
12.2 Separation of variables . . . 27
12.3 The K-Bessel Fourier expansion . . . 27
12.4 Hecke operators and Fourier–Hecke relations . . . 27
12.5 Parity, spectral parameters, and normalizations . . . 27

13 Eisenstein Series **28**
13.1 Construction and initial convergence . . . 28
13.2 Fourier expansion and the constant term . . . 28
13.3 The scattering coefficient and zeta factors . . . 28
13.4 Continuation, functional equation, and residues . . . 28
13.5 Incomplete Eisenstein series . . . 28

14 Truncation and the Maass–Selberg Relations **29**
14.1 Two forms of truncation . . . 29
14.2 Boundary terms from Green identities . . . 29
14.3 The Maass–Selberg calculation . . . 29
14.4 Coincident parameters and limiting cases . . . 29
14.5 Scattering identities and spectral measure . . . 29

15 The Spectral Decomposition of the Modular Surface **30**
15.1 The discrete spectrum and the constant eigenfunction . . . 30
15.2 The cuspidal subspace . . . 30
15.3 The Eisenstein contribution . . . 30
15.4 Completeness and spectral expansion . . . 30
15.5 Parseval, Plancherel, and modes of convergence . . . 30

16 L-Functions, Rankin–Selberg Integrals, and Spectral Weights **31**
16.1 Normalized L-functions and Euler/Hecke conventions . . . 31
16.2 Holomorphic Mellin integrals and functional equations . . . 31
16.3 Maass Mellin integrals and parity . . . 31
16.4 Rankin–Selberg unfolding and residues . . . 31
16.5 Norm identities and harmonic spectral weights . . . 31

IV Kernels and Selberg **32**

17 Automorphic Kernels and the Pre-Trace Formula **33**
17.1 Point-pair invariants and radial kernels . . . 33
17.2 The Selberg transform and its inverse . . . 33
17.3 Periodization and convergence . . . 33
17.4 Action on discrete and Eisenstein eigenfunctions . . . 33
17.5 The pre-trace identity . . . 33

18 Conjugacy Classes, Closed Geodesics, and Orbital Integrals **34**
18.1 Conjugacy classes and centralizers . . . 34
18.2 Orbital unfolding . . . 34
18.3 Identity and elliptic terms . . . 34
18.4 Primitive closed geodesics and hyperbolic integrals . . . 34
18.5 Parabolic classes and cusp geometry . . . 34

19 The Selberg Trace Formula **35**
19.1 Why the naive diagonal trace diverges . . . 35
19.2 Simultaneous geometric and spectral truncation . . . 35
19.3 Cancellation and the limiting argument . . . 35
19.4 The complete modular-group formula . . . 35
19.5 Test functions and equivalent versions . . . 35

20 Applications of Selberg's Formula **36**
20.1 Designing spectral test functions . . . 36
20.2 Weyl law and scattering contributions . . . 36
20.3 The Selberg zeta function . . . 36
20.4 A first prime geodesic theorem . . . 36
20.5 Smoothing, unsmoothing, and stronger estimates . . . 36

V Fourier periods and Kuznetsov **37**

21 Fourier Periods and the Pre-Kuznetsov Formula **38**
21.1 Fourier coefficients as periods . . . 38
21.2 Poincaré series with controlled seeds . . . 38
21.3 Unfolded spectral pairings . . . 38
21.4 Unipotent double cosets and Kloosterman sums . . . 38
21.5 The preliminary relative trace identities . . . 38

22 The Bessel Transforms of Kuznetsov **39**
22.1 Same-sign kernels . . . 39
22.2 Opposite-sign kernels . . . 39
22.3 Transform inversion and admissibility . . . 39
22.4 Contour shifts, residues, and holomorphic terms . . . 39
22.5 Decay and differentiation estimates . . . 39

23 The Kuznetsov Trace Formulas **40**
23.1 The spectral-to-arithmetic statement . . . 40
23.2 The arithmetic-to-spectral statement . . . 40
23.3 Signs, parity, continuous and holomorphic terms . . . 40
23.4 Fourier and Hecke normalization dictionaries . . . 40
23.5 Orthogonality, spectral moments, and Kloosterman averages . . . 40

VI Classical relative trace formulas **41**

24 Periods, Double Cosets, and Closed-Geodesic Trace Formulas **42**
24.1 Period functionals and smoothing kernels . . . 42
24.2 Relative unfolding and double cosets . . . 42
24.3 Fourier periods revisited . . . 42
24.4 A closed-geodesic relative trace formula . . . 42
24.5 Stabilizers, orbital integrals, and the continuous spectrum . . . 42

25 Relative Trace Formulas for Central L-Values **43**
25.1 Mellin periods and functional-equation signs . . . 43
25.2 Double periods of the holomorphic kernel . . . 43
25.3 The geometric decomposition and regularization . . . 43
25.4 Hecke insertions and weighted moments . . . 43
25.5 An explicit formula and a first application . . . 43

VII Weighted trace formulas **44**

26 Weighted Diagonal Integrals and Spectral Matrix Elements **45**
26.1 Compactly supported weights . . . 45
26.2 Spectral expansion of the weighted diagonal . . . 45
26.3 Multiplication operators and trace-class hypotheses . . . 45
26.4 Weighted traces versus period pairings . . . 45
26.5 First localized spectral averages . . . 45

27 Eigenfunction Weights and Geodesic Periods **46**
27.1 Cuspidal Laplace eigenfunctions as weights . . . 46
27.2 Weighted orbital integrals . . . 46
27.3 Averaging the eigenfunction equation . . . 46
27.4 Geodesic periods and eigenvalue-dependent transforms . . . 46
27.5 The full cuspidal-weight formula . . . 46

28 Regularization and the Zelditch–Biró Formulas **47**
28.1 Noncuspidal weights and their constant terms . . . 47
28.2 Weighted truncation and regularized pairings . . . 47
28.3 Poles, exceptional parameters, and boundary contributions . . . 47
28.4 A normalization and transform dictionary . . . 47
28.5 The exact common scope of the comparison . . . 47

VIII Connections and synthesis **48**

29 Classical Weight-k Operators and Hecke-Twisted Trace Formulas **49**
29.1 Unitary weight-k automorphy 49
29.2 Weight-k Laplacians 49
29.3 Raising and lowering operators 49
29.4 The holomorphic contribution in the correct weighted space 49
29.5 Hecke-twisted kernels and trace comparisons 49

30 A Dictionary Between Trace Formulas **50**
30.1 A common inventory of kernels and functionals 50
30.2 Direct specializations 50
30.3 Common-kernel derivations 50
30.4 Comparisons requiring weights, projections, or Hecke operators 50
30.5 Information retained and discarded; the bridge to Volume 1b 50

A Analytic Foundations **51**

B Special Functions **52**

C Functional Analysis and Differential Operators **53**

D Integral Transforms **54**

E Arithmetic Estimates and Summation Tools **55**

F Period Polynomials **56**

G Normalization Dictionary **57**

H Supplementary Proofs **58**

I Notational Index **59**

Part I

Foundations

Chapter 0

What Is a Trace Formula?

This book is concerned with the concept trace formulas with a focus on trace formulas of automorphic forms on $\mathrm{SL}_2(\mathbb{Z})$. In general a trace formula is a formula where the trace of some operator is computed in two different ways leading to an equality. We call these the spectral side and geometric side of the trace formula.

This chapter establishes the elementary identities and conventions that underlie that comparison. Section 0.1 treats matrix traces. Section 0.2 passes from orthonormal eigenfunctions to an integral kernel, derives the diagonal trace identity, and extends it to a general matrix. Section 0.3 distinguishes the trace of a kernel from a relative pairing and a weighted diagonal integral.

0.1 Finite matrices and two descriptions of a trace

We begin with the simplest example of a trace formula. Consider the complex matrix

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

We compute its trace in two ways. The first is by adding its diagonal entries to get 4. We regard this coordinate expression as the geometric side of this elementary trace formula: a choice of basis expresses the linear transformation as a matrix, whose diagonal entries we sum.

The second way is to add up the eigenvalues. The vectors $(1,1)^{\mathsf{T}}$ and $(1,-1)^{\mathsf{T}}$ are eigenvectors with eigenvalues 3 and 1, so the eigenvalues also add to 4. This is the spectral side. Thus, the trace formula in this example is simply $4 = 4$.

The goal of this section is to turn this example into the following theorem from a first course in linear algebra.

Theorem 0.1.1 (Trace as a sum of eigenvalues)**.** *Let $A = (a_{ij}) \in M_n(\mathbb{C})$, where $n \geq 0$. List the roots $\lambda_1, \ldots, \lambda_n$ of its characteristic polynomial $\det(tI_n - A)$, repeating each root according to its multiplicity. Then*

$$\sum_{i=1}^{n} a_{ii} = \sum_{j=1}^{n} \lambda_j.$$

For $n = 0$, both sums are empty and equal to zero.

The multiplicity in this statement is **algebraic multiplicity**: the multiplicity as a polynomial root. It is not the dimension of the corresponding eigenspace. A complex number λ is such a root exactly when $\lambda I_n - A$ is singular, equivalently when there exists a nonzero vector v with $Av = \lambda v$.

We will prove the theorem after developing the basic properties of the trace.

Definition 0.1.2 (Trace)**.** For an $n \times n$ matrix $A = (a_{ij})$, define

$$\operatorname{tr}(A) = \sum_{i=1}^{n} a_{ii}.$$

For $n = 0$ this is the empty sum, equal to zero.

The definition immediately gives the following properties, for $A, B \in M_n(\mathbb{C})$ and $c \in \mathbb{C}$:

- $\operatorname{tr}(A + B) = \operatorname{tr}(A) + \operatorname{tr}(B)$
- $\operatorname{tr}(cA) = c \operatorname{tr}(A)$
- $\operatorname{tr}(I_n) = n$

While matrix multiplication is not commutative, reversing the order of two compatible factors leaves the trace unchanged. The two products need not even have the same size.

Example 0.1.3 (Rectangular matrices)**.** Let $A = (1\ 2)$ and $B = (3\ 4)^{\mathsf{T}}$. Then $AB = (11)$ and

$$BA = \begin{pmatrix} 3 & 6 \\ 4 & 8 \end{pmatrix}.$$

The traces are 11 and $3 + 8 = 11$, although the matrices have different dimensions.

This equality holds for every pair of compatible rectangular matrices.

Proposition 0.1.4 (Cyclicity for two factors)**.** *Let A be an $m \times n$ complex matrix and B an $n \times m$ complex matrix, where $m, n \geq 0$. Then*

$$\operatorname{tr}(AB) = \operatorname{tr}(BA).$$

Proof. By the definition of matrix multiplication,

$$\begin{aligned} \operatorname{tr}(AB) &= \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} b_{ji} \\ &= \sum_{j=1}^{n} \sum_{i=1}^{m} b_{ji} a_{ij} = \operatorname{tr}(BA). \end{aligned}$$

The middle equality interchanges two finite sums and commutes two complex numbers. □

We can now prove that the trace does not depend on the basis chosen to represent a linear transformation. A change of basis changes its matrix by conjugation, and cyclicity shows that this leaves the trace unchanged.

Corollary 0.1.5 (Invariance under a change of basis)**.** *For $A \in M_n(\mathbb{C})$ and invertible $S \in M_n(\mathbb{C})$,*

$$\operatorname{tr}(S^{-1}AS) = \operatorname{tr}(A).$$

Proof. Apply cyclicity to the two factors $S^{-1}A$ and S. Associativity and $SS^{-1} = I_n$ give

$$\operatorname{tr}((S^{-1}A)S) = \operatorname{tr}(S(S^{-1}A)) = \operatorname{tr}(A).$$

□

Definition 0.1.6 (Trace of an endomorphism)**.** If T is a linear map from a finite-dimensional complex vector space to itself, its matrices in two bases are related by this change of basis. We can therefore define $\operatorname{tr}(T)$ using any basis: the answer is independent of that choice.

We return to the trace–eigenvalue theorem stated at the beginning. In the notation just introduced, it says $\operatorname{tr}(A) = \sum_{j=1}^{n} \lambda_j$.

Proof of the trace–eigenvalue theorem. The case $n = 0$ says $0 = 0$. Assume $n \geq 1$. We compute the coefficient of t^{n-1} in the characteristic polynomial in two ways.

In the determinant expansion, the identity permutation contributes

$$\prod_{i=1}^{n}(t - a_{ii}),$$

whose t^{n-1} coefficient is $-\sum_i a_{ii}$. Every nonidentity permutation moves at least two indices. At each moved index its determinant factor is an off-diagonal entry of $tI_n - A$, which is independent of t. Such a term has degree at most $n - 2$ and contributes nothing to the coefficient in question. For $n = 1$ there are no nonidentity permutations. Thus the coefficient is $-\operatorname{tr}(A)$.

By the fundamental theorem of algebra and monicity,

$$\det(tI_n - A) = \prod_{j=1}^{n}(t - \lambda_j).$$

The coefficient of t^{n-1} in this product is $-\sum_j \lambda_j$. Equating the two coefficients proves the theorem. □

For a diagonalizable matrix, the theorem also follows from change-of-basis invariance: choose a basis of eigenvectors, so that the diagonal entries are the eigenvalues. The strength of the coefficient proof is that it applies even when the matrix is not diagonalizable. It counts roots of the characteristic polynomial with their algebraic multiplicities, even when there are too few independent eigenvectors to form a basis. The following example illustrates this distinction.

Example 0.1.7 (A non-diagonalizable matrix)**.** Take

$$J = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}.$$

Its characteristic polynomial is $(t-2)^2$, while $(J - 2I_2)(x, y) = (y, 0)$. Its eigenspace consists of $(x, 0)$ and is only one-dimensional. Nevertheless the theorem gives $\operatorname{tr}(J) = 2 + 2 = 4$.

0.2 Finite-rank kernels and diagonal integration

The spectral side of a trace formula records how an operator acts on eigenfunctions. In a finite-dimensional setting, the passage from this spectral description to a kernel follows three steps:

1. Choose an orthonormal basis of eigenfunctions for a finite-dimensional invariant subspace.
2. Take inner products with those eigenfunctions to extract coefficients and write the action of the operator.
3. Express that action through an integral kernel, then integrate the kernel along the diagonal to recover the trace.

We carry out this procedure for a finite family of eigenfunctions. We then recover the more general matrix construction, which also applies when an orthonormal eigenbasis is unavailable.

Let (X, μ) be a sigma-finite measure space and put $H = L^2(X, \mu; \mathbb{C})$, with inner product and norm

$$\langle f, g\rangle_{L^2(X,\mu)} = \int_X f(x)\overline{g(x)}\, d\mu(x),$$
$$\|f\|_{L^2(X,\mu)} = \left(\int_X |f(x)|^2\, d\mu(x)\right)^{1/2}.$$

Fix an orthonormal family $u_1, \dots, u_N$, choose measurable representatives, and let V be its span. Suppose $T : H \to H$ is linear, satisfies $Tu_j = \lambda_j u_j$ with $\lambda_j \in \mathbb{C}$, and vanishes on $V^\perp$. Thus the u_j form an orthonormal eigenbasis of V; they need not span all of H.

To determine Tf, we first extract the part of f lying in V.

Proposition 0.2.1 (Coefficients in an orthonormal family)**.** *Every $f \in H$ has a unique decomposition*

$$f = \sum_{j=1}^{N} \langle f, u_j\rangle_{L^2(X,\mu)} u_j + r, \qquad r \in V^\perp.$$

Consequently,

$$Tf = \sum_{j=1}^{N} \lambda_j \langle f, u_j\rangle_{L^2(X,\mu)} u_j.$$

Proof. Subtract the displayed finite sum from f to define r. For each k, linearity in the first argument and orthonormality give

$$\begin{aligned}\langle r, u_k\rangle_{L^2(X,\mu)} &= \langle f, u_k\rangle_{L^2(X,\mu)} - \sum_{j=1}^{N} \langle f, u_j\rangle_{L^2(X,\mu)} \langle u_j, u_k\rangle_{L^2(X,\mu)} \\ &= 0.\end{aligned}$$

It follows that r is orthogonal to every vector in V. Any decomposition of f as a linear combination of the u_j plus a vector in $V^\perp$ has these same coefficients, as taking the inner product with u_k shows. This proves uniqueness. Applying T, using $Tr = 0$ and $Tu_j = \lambda_j u_j$, proves the operator formula. □

Cauchy–Schwarz gives $|\langle f, u_j\rangle_{L^2(X,\mu)}| \le \|f\|_{L^2(X,\mu)}$. Hence the coefficient formula implies

$$\|Tf\|_{L^2(X,\mu)} \le \left(\sum_{j=1}^{N} |\lambda_j|\right) \|f\|_{L^2(X,\mu)}.$$

In particular, T is bounded and has image in V.

We now substitute the integral defining each coefficient into the operator formula. The finite sum can be brought inside the integral, suggesting the following kernel.

Definition 0.2.2 (Kernel from an orthonormal eigenbasis)**.** Using the chosen representatives of the u_j, define

$$K_T(x,y) = \sum_{j=1}^{N} \lambda_j u_j(x)\overline{u_j(y)}.$$

For $f \in H$, Cauchy–Schwarz gives

$$\int_X |u_j(y)f(y)|\, d\mu(y) \le \|u_j\|_{L^2(X,\mu)}\|f\|_{L^2(X,\mu)} = \|f\|_{L^2(X,\mu)}.$$

Each term is therefore integrable in y, and termwise integration yields

$$\begin{aligned}\int_X K_T(x,y)f(y)\, d\mu(y) &= \sum_{j=1}^{N} \lambda_j u_j(x) \int_X f(y)\overline{u_j(y)}\, d\mu(y)\\ &= (Tf)(x)\end{aligned}$$

for almost every x. This identifies K_T as an integral kernel for T.

To compare its diagonal integral with a trace, we extend the finite-dimensional definition from Section 0.1 to the operators used here.

Definition 0.2.3 (Finite-rank trace)**.** If a linear operator $S : H \to H$ has image in a finite-dimensional subspace W and vanishes on $W^{\perp}$, define

$$\operatorname{tr}(S) = \operatorname{tr}(S|_W).$$

Well-definedness. The definition uses an auxiliary choice of W, so we must show that the resulting number depends only on S. For a fixed W, the restriction $S|_W$ is an endomorphism because $S(H) \subseteq W$, and its trace is independent of the basis by Corollary 0.1.5.

Now let W and W' be two possible choices in the definition: each is finite-dimensional, each contains the image of S, and S vanishes on each one's orthogonal complement. We must prove

$$\operatorname{tr}(S|_W) = \operatorname{tr}(S|_{W'}).$$

The spaces need not contain one another. To compare them, put $U = W + W'$, a finite-dimensional space containing both. Since $W \subseteq U$, we have the orthogonal decomposition

$$U = W \oplus (U \cap W^{\perp}).$$

The operator maps W into itself and is zero on the second summand. Thus a basis adapted to this decomposition gives the matrix

$$\begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}$$

for $S|_U$, where B is a matrix of $S|_W$. Adding the zero block does not change the sum of the diagonal entries, so $\operatorname{tr}(S|_U) = \operatorname{tr}(S|_W)$. Applying the same argument to $U = W' \oplus (U \cap (W')^{\perp})$ gives

$$\operatorname{tr}(S|_W) = \operatorname{tr}(S|_U) = \operatorname{tr}(S|_{W'}).$$

Consequently the definition of $\operatorname{tr}(S)$ is independent of the choice of W. □

For the operator T, its matrix on V in the basis $u_1, \dots, u_N$ is $\operatorname{diag}(\lambda_1, \dots, \lambda_N)$. Thus its trace is $\sum_j \lambda_j$. On the kernel side,

$$K_T(x,x) = \sum_{j=1}^{N} \lambda_j |u_j(x)|^2.$$

Each summand is integrable, and the normalization $\|u_j\|_{L^2(X,\mu)} = 1$ gives

$$\int_X K_T(x,x)\, d\mu(x) = \sum_{j=1}^{N} \lambda_j \int_X |u_j(x)|^2\, d\mu(x) = \sum_{j=1}^{N} \lambda_j = \operatorname{tr}(T).$$

This is the trace formula produced by the three steps: the diagonal integral is the kernel expression, and the sum of eigenvalues is the spectral expression.

Example 0.2.4 (A rank-one kernel)**.** On $[0,1]$ with Lebesgue measure dx, take $u(x) = \sqrt{3}\, x$. Since

$$\|u\|^2_{L^2([0,1],dx)} = 3 \int_0^1 x^2\, dx = 1,$$

for any $\alpha \in \mathbb{C}$ we may prescribe $Tu = \alpha u$ and let T vanish on $u^{\perp}$. The coefficient formula gives

$$K_T(x,y) = 3\alpha x y, \qquad (Tf)(x) = 3\alpha x \int_0^1 y f(y)\, dy.$$

The two trace calculations agree:

$$\int_0^1 K_T(x,x)\, dx = 3\alpha \int_0^1 x^2\, dx = \alpha = \operatorname{tr}(T).$$

The eigenbasis calculation has a general matrix form. Retain the orthonormal family and its span V, and now allow an arbitrary matrix $A \in M_N(\mathbb{C})$ to describe the action on V.

Definition 0.2.5 (Finite kernel and associated operator)**.** For $A = (a_{ij}) \in M_N(\mathbb{C})$, define

$$T_A f = \sum_{i,j=1}^{N} a_{ij} \langle f, u_j \rangle_{L^2(X,\mu)} u_i,$$

$$K_A(x,y) = \sum_{i,j=1}^{N} a_{ij} u_i(x) \overline{u_j(y)}.$$

The same Cauchy–Schwarz estimate makes each integral against $\overline{u_j(y)} f(y)$ absolutely convergent, so

$$(T_A f)(x) = \int_X K_A(x,y) f(y)\, d\mu(y)$$

for almost every x. Moreover,

$$\|T_A f\|_{L^2(X,\mu)} \le \left(\sum_{i,j} |a_{ij}| \right) \|f\|_{L^2(X,\mu)}.$$

Thus T_A is bounded, has image in V, and vanishes on $V^\perp$. Since $T_A u_j = \sum_i a_{ij} u_i$, its restriction to V has matrix A, and $\operatorname{tr}(T_A) = \operatorname{tr}(A)$. The diagonal choice $A = \operatorname{diag}(\lambda_1, \dots, \lambda_N)$ recovers T and K_T.

For a finite set $X = \{1, \dots, N\}$ with counting measure and the point indicators as basis, $K_A(i,j) = a_{ij}$. The kernel formula then reduces to matrix multiplication. In general, its two variables play the roles of the output and input indices.

Proposition 0.2.6 (Finite-rank trace identity)**.** *The specified diagonal $x \mapsto K_A(x,x)$ is integrable. If $\nu_1, \dots, \nu_N$ are the eigenvalues of A, counted with algebraic multiplicity, then*

$$\boxed{\int_X K_A(x,x)\, d\mu(x) = \operatorname{tr}(T_A) = \sum_{j=1}^{N} \nu_j.}$$

Proof. For every i, j,

$$\int_X |u_i(x)\overline{u_j(x)}|\, d\mu(x) \le \|u_i\|_{L^2(X,\mu)} \|u_j\|_{L^2(X,\mu)} = 1.$$

The diagonal is therefore a finite sum of integrable functions. Orthonormality gives

$$\int_X K_A(x,x)\, d\mu(x) = \sum_{i,j} a_{ij} \langle u_i, u_j \rangle_{L^2(X,\mu)} = \sum_i a_{ii} = \operatorname{tr}(A).$$

The matrix of $T_A|_V$ is A, and the trace–eigenvalue identity supplies the last equality. □

The numbers ν_j are eigenvalues of the restriction $T_A|_V$. No diagonalizability assumption is needed, and the chosen orthonormal family need not consist of eigenfunctions of T_A. Taking $A = I_N$ gives orthogonal projection onto V, whose trace is N. The general matrix form will be used for the relative pairings and weighted traces in Section 0.3.

Remark 0.2.7 (The diagonal specified by the kernel formula). The trace identity uses the kernel constructed from the chosen representatives of the u_j:

$$K_A(x,y) = \sum_{i,j=1}^{N} a_{ij} u_i(x) \overline{u_j(y)}.$$

We may change each u_j arbitrarily on a μ-null set E_j, provided the new function $\widetilde{u}_j$ is measurable. Rebuilding the kernel from these new representatives gives

$$\widetilde{K}_A(x,y) = \sum_{i,j=1}^{N} a_{ij} \widetilde{u}_i(x) \overline{\widetilde{u}_j(y)}.$$

Put $E = \bigcup_{j=1}^{N} E_j$, which is again μ-null. The kernels K_A and $\widetilde{K}_A$ agree outside

$$(E \times X) \cup (X \times E),$$

a null set for the product measure $\mu \otimes \mu$ on $X \times X$. Moreover, $\widetilde{K_A}(x,x) = K_A(x,x)$ whenever $x \notin E$, so their diagonal integrals agree. Thus changing the representatives of the u_j changes the kernel only on a product-null set, and changes its diagonal only on a μ-null set.

We cannot make an arbitrary measurable change to K_A on a $(\mu \otimes \mu)$-null set and still assert the diagonal trace identity. Such a change preserves the integral operator almost everywhere, but a product-null set can contain the entire diagonal. For example, the zero function and the indicator of $\{(x,y) : x = y\}$ agree almost everywhere on $[0,1]^2$ for two-dimensional Lebesgue measure. Both define the zero integral operator, but their diagonal integrals are respectively zero and one. The kernel supplied by our finite formula for the zero operator is the zero function. Replacing it by the diagonal indicator preserves the operator but abandons the formula that specifies its diagonal. The distinction is therefore between changes induced by choosing representatives of the u_j in the finite formula, which preserve the diagonal integral, and arbitrary product-null changes to the kernel, which need not.

0.3 Ordinary traces, relative pairings, and weights

Retain the measure space (X, μ), the orthonormal family $u_1, \dots, u_N$, its span V, and the kernel K_A from Section 0.2. There are three distinct operations: integrate the diagonal, integrate against separate functions in the two variables, or weight the diagonal before integrating. Each is linear in K_A.

Definition 0.3.1 (Ordinary diagonal trace)**.** The ordinary diagonal trace is

$$\mathcal{T}(K_A) = \int_X K_A(x,x)\, d\mu(x).$$

The finite-rank trace identity gives $\mathcal{T}(K_A) = \operatorname{tr}(A)$. If $A = \operatorname{diag}(\lambda_1, \dots, \lambda_N)$, this is $\sum_j \lambda_j$.

We now consider a relative pairing, which measures the operator against a chosen input and a chosen output test. Choose a function $q \in H$ and apply T_A to it. We can then test the output against another function $p \in H$ by taking $\langle T_A q, p\rangle_{L^2(X,\mu)}$. In the kernel integral, $q(y)$ specifies the input and $\overline{p(x)}$ tests the output. When A is diagonal, each eigenvalue is accompanied by two coefficients: one measuring the component of q along its eigenfunction, and one measuring that eigenfunction against p. This leads to the following definition.

Definition 0.3.2 (Periods and relative pairings)**.** For $u, p \in H = L^2(X, \mu; \mathbb{C})$, write

$$P_p(u) = \langle u, p\rangle_{L^2(X,\mu)} = \int_X u(x)\overline{p(x)}\, d\mu(x).$$

Given $p, q \in H$, define

$$\mathcal{R}_{p,q}(K_A) = \int_X \int_X K_A(x,y) q(y) \overline{p(x)}\, d\mu(y)\, d\mu(x).$$

The pairing is called relative here because it depends on the two chosen test functions p and q. We use the same kernel K_A, integrated against these functions in its two variables. The functional P_p is an elementary model of a period: it integrates u against a fixed function.

Proposition 0.3.3 (Relative kernel identity)**.** *The integral is absolutely convergent and satisfies*

$$\mathcal{R}_{p,q}(K_A) = \langle T_A q, p\rangle_{L^2(X,\mu)} = \sum_{i,j} a_{ij} P_p(u_i) \overline{P_q(u_j)}.$$

Proof. The finite kernel formula separates the dependence on x and y:

$$K_A(x,y)q(y)\overline{p(x)} = \sum_{i,j=1}^{N} a_{ij}\bigl(u_i(x)\overline{p(x)}\bigr)\bigl(q(y)\overline{u_j(y)}\bigr).$$

We first check absolute convergence, so that we can integrate these summands separately.

For each i, Cauchy–Schwarz gives

$$\begin{aligned}\int_X |u_i(x)\overline{p(x)}|\,d\mu(x) &\le \|u_i\|_{L^2(X,\mu)}\|p\|_{L^2(X,\mu)}\\ &= \|p\|_{L^2(X,\mu)}.\end{aligned}$$

The last equality uses $\|u_i\|_{L^2(X,\mu)} = 1$, since the family is orthonormal. Similarly,

$$\int_X |q(y)\overline{u_j(y)}|\,d\mu(y) \le \|q\|_{L^2(X,\mu)}.$$

Thus the two factors in each summand are integrable in their respective variables. Tonelli's theorem applies to their nonnegative absolute values and gives

$$\begin{aligned}&\int_X\int_X |u_i(x)\overline{p(x)}|\,|q(y)\overline{u_j(y)}|\,d\mu(y)\,d\mu(x)\\ &\quad= \left(\int_X |u_i(x)\overline{p(x)}|\,d\mu(x)\right)\left(\int_X |q(y)\overline{u_j(y)}|\,d\mu(y)\right)\\ &\quad\le \|p\|_{L^2(X,\mu)}\|q\|_{L^2(X,\mu)}.\end{aligned}$$

Multiplying by $|a_{ij}|$, summing over the finitely many pairs (i,j), and applying the triangle inequality now yields

$$\begin{aligned}&\int_X\int_X |K_A(x,y)q(y)\overline{p(x)}|\,d\mu(y)\,d\mu(x)\\ &\quad\le \left(\sum_{i,j=1}^{N}|a_{ij}|\right)\|p\|_{L^2(X,\mu)}\|q\|_{L^2(X,\mu)} < \infty.\end{aligned}$$

This proves the asserted absolute convergence.

We can now evaluate the integral. For fixed x, the factor $u_i(x)\overline{p(x)}$ is constant with respect to y, so it comes outside the inner integral. The remaining y-integral is a scalar independent of x. Fubini's theorem and linearity of integration therefore give

$$\mathcal{R}_{p,q}(K_A) = \sum_{i,j=1}^{N} a_{ij}\left(\int_X u_i(x)\overline{p(x)}\,d\mu(x)\right)\left(\int_X q(y)\overline{u_j(y)}\,d\mu(y)\right).$$

The first integral is $P_p(u_i)$ by definition. For the second, conjugation commutes with the integral of an integrable complex-valued function, so

$$\begin{aligned}\int_X q(y)\overline{u_j(y)}\,d\mu(y) &= \overline{\int_X u_j(y)\overline{q(y)}\,d\mu(y)}\\ &= \overline{P_q(u_j)}.\end{aligned}$$

This explains the conjugate on the second period and proves

$$\mathcal{R}_{p,q}(K_A) = \sum_{i,j=1}^{N} a_{ij} P_p(u_i)\overline{P_q(u_j)}.$$

Finally, the kernel representation of T_A established in Section 0.2 says that

$$(T_A q)(x) = \int_X K_A(x,y) q(y)\, d\mu(y)$$

for almost every x. Substituting this into the defining integral for the relative pairing gives

$$\begin{aligned}\mathcal{R}_{p,q}(K_A) &= \int_X (T_A q)(x)\overline{p(x)}\, d\mu(x) \\ &= \langle T_A q, p\rangle_{L^2(X,\mu)}.\end{aligned}$$

The almost-everywhere equality suffices because changing an integrable function on a μ-null set does not change its integral. This establishes the remaining equality. □

Corollary 0.3.4 (Relative pairing in an eigenbasis)**.** *If $A = \operatorname{diag}(\lambda_1, \dots, \lambda_N)$, then for every $p, q \in H$,*

$$\mathcal{R}_{p,q}(K_A) = \sum_{j=1}^{N} \lambda_j P_p(u_j)\overline{P_q(u_j)}.$$

Proof. In the relative kernel identity, $a_{ij} = 0$ when $i \neq j$ and $a_{jj} = \lambda_j$. Only the terms with $i = j$ remain, giving the stated sum. □

Thus separate measurements of the eigenfunctions remain in the formula.

We now insert a function $w(x)$ to weight the diagonal contribution at each point before integrating.

Definition 0.3.5 (Weighted diagonal integral)**.** Let $w \in L^\infty(X, \mu; \mathbb{C})$, let $M_w f = wf$, and let P be orthogonal projection onto V. Define

$$\mathcal{W}_w(K_A) = \int_X w(x) K_A(x,x)\, d\mu(x).$$

Essential boundedness of w and integrability of the specified diagonal make this integral absolutely convergent.

Proposition 0.3.6 (Weighted kernel identity)**.** *For the endomorphism $S_w = P M_w T_A|_V$,*

$$\mathcal{W}_w(K_A) = \operatorname{tr}(S_w) = \sum_{i,j} a_{ij} \int_X w(x) u_i(x)\overline{u_j(x)}\, d\mu(x).$$

Proof. We first explain why S_w is an endomorphism of V. Essential boundedness means that $|w(x)| \leq \|w\|_{L^\infty(X,\mu)}$ for almost every x. Consequently, for every $f \in H$,

$$\|wf\|_{L^2(X,\mu)} \leq \|w\|_{L^\infty(X,\mu)} \|f\|_{L^2(X,\mu)},$$

so multiplication by w maps H to H. For $v \in V$, we can therefore form $wT_A v \in H$ and project it onto V. This gives the linear map

$$S_w v = P(wT_A v) \in V.$$

The projection ensures that the output lies in V, even if multiplication by w takes it outside V.

We next explain why the projection can be removed when computing a diagonal entry. For any $h \in H$, the definition of orthogonal projection gives $h - Ph \in V^\perp$. Since $u_j \in V$, it follows that

$$\begin{aligned}\langle Ph, u_j\rangle_{L^2(X,\mu)} &= \langle h, u_j\rangle_{L^2(X,\mu)} - \langle h - Ph, u_j\rangle_{L^2(X,\mu)} \\ &= \langle h, u_j\rangle_{L^2(X,\mu)}.\end{aligned}$$

In the orthonormal basis $u_1, \dots, u_N$, the j-th diagonal entry of S_w is the coefficient of u_j in $S_w u_j$, namely $\langle S_w u_j, u_j\rangle_{L^2(X,\mu)}$. Also, $T_A|_V$ has matrix A, so $T_A u_j = \sum_{i=1}^N a_{ij} u_i$. Applying the preceding projection identity with $h = wT_A u_j$ gives

$$\begin{aligned}\langle S_w u_j, u_j\rangle_{L^2(X,\mu)} &= \langle wT_A u_j, u_j\rangle_{L^2(X,\mu)} \\ &= \sum_{i=1}^N a_{ij} \langle w u_i, u_j\rangle_{L^2(X,\mu)} \\ &= \sum_{i=1}^N a_{ij} \int_X w(x) u_i(x) \overline{u_j(x)}\, d\mu(x).\end{aligned}$$

The coefficients a_{ij} come outside unchanged because the inner product is linear in its first argument. Summing these diagonal entries gives

$$\operatorname{tr}(S_w) = \sum_{i,j=1}^N a_{ij} \int_X w(x) u_i(x) \overline{u_j(x)}\, d\mu(x).$$

Finally, we obtain the same sum from the weighted kernel. Each term is integrable: essential boundedness and Cauchy–Schwarz give

$$\begin{aligned}\int_X |w(x) u_i(x) \overline{u_j(x)}|\, d\mu(x) &\le \|w\|_{L^\infty(X,\mu)} \|u_i\|_{L^2(X,\mu)} \|u_j\|_{L^2(X,\mu)} \\ &= \|w\|_{L^\infty(X,\mu)},\end{aligned}$$

where both basis functions have norm 1. We may therefore integrate the finite diagonal formula term by term:

$$\begin{aligned}\mathcal{W}_w(K_A) &= \int_X w(x) \sum_{i,j=1}^N a_{ij} u_i(x) \overline{u_j(x)}\, d\mu(x) \\ &= \sum_{i,j=1}^N a_{ij} \int_X w(x) u_i(x) \overline{u_j(x)}\, d\mu(x) \\ &= \operatorname{tr}(S_w).\end{aligned}$$

□

Corollary 0.3.7 (Weighted diagonal in an eigenbasis)**.** *If* $A = \operatorname{diag}(\lambda_1, \dots, \lambda_N)$ *and* $w \in L^\infty(X, \mu; \mathbb{C})$*, then*

$$\mathcal{W}_w(K_A) = \sum_{j=1}^{N} \lambda_j \int_X w(x)|u_j(x)|^2 \, d\mu(x).$$

Proof. In the weighted kernel identity, all terms with $i \neq j$ vanish. For $i = j$, substitute $a_{jj} = \lambda_j$ and $u_j(x)\overline{u_j(x)} = |u_j(x)|^2$. □

Example 0.3.8 (Three measurements of the same kernels)**.** Let $X = \{1, 2\}$ with counting measure μ, and take the indicator basis $u_1 = 1_{\{1\}}$, $u_2 = 1_{\{2\}}$ of $H = L^2(X, \mu; \mathbb{C})$. Then $V = H$. Consider the matrices

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

These define kernels $K_A(r, s) = a_{rs}$ and $K_B(r, s) = b_{rs}$. All integrals in this example are sums over the indicated points.

Ordinary trace. Integrating the diagonal gives

$$\mathcal{T}(K_A) = K_A(1, 1) + K_A(2, 2) = 1 + 2 = 3,$$

and likewise $\mathcal{T}(K_B) = 3$. This calculation uses the two diagonal entries and agrees with the matrix trace.

Relative pairing. Choose $p = u_1$ and $q = u_2$. These choices give

$$P_p(u_1) = 1, \qquad P_p(u_2) = 0, \qquad P_q(u_1) = 0, \qquad P_q(u_2) = 1.$$

The relative kernel identity therefore selects the entry in row one and column two:

$$\mathcal{R}_{u_1,u_2}(K_A) = a_{12} = 0, \qquad \mathcal{R}_{u_1,u_2}(K_B) = b_{12} = 1.$$

Directly in the kernel integral, $q(y)$ restricts the input variable to $y = 2$ and $\overline{p(x)}$ restricts the output variable to $x = 1$. In operator terms,

$$T_A u_2 = (0, 2)^{\mathsf{T}}, \qquad T_B u_2 = (1, 2)^{\mathsf{T}},$$

and pairing with u_1 extracts the first coordinate. Thus this relative pairing distinguishes the two kernels even though their ordinary traces agree.

Weighted diagonal. Write $w_1 = w(1)$ and $w_2 = w(2)$ for an arbitrary complex weight. Then

$$\mathcal{W}_w(K_A) = w_1 K_A(1, 1) + w_2 K_A(2, 2) = w_1 + 2w_2,$$

and the same calculation gives $\mathcal{W}_w(K_B) = w_1 + 2w_2$. Since $V = H$, the projection P is the identity. Multiplication by w has matrix $D_w = \operatorname{diag}(w_1, w_2)$, so the two weighted endomorphisms have matrices

$$D_w A = \begin{pmatrix} w_1 & 0 \\ 0 & 2w_2 \end{pmatrix}, \qquad D_w B = \begin{pmatrix} w_1 & w_1 \\ w_2 & 2w_2 \end{pmatrix}.$$

Their traces are exactly the weighted diagonal integrals above. For $w = 1_{\{1\}}$, both integrals equal 1; for $w = 1_{\{2\}}$, both equal 2; and $w = 1$ recovers the ordinary trace 3. These weights distinguish the contributions of the two points, but no diagonal weight distinguishes K_A from K_B. Their difference lies in the off-diagonal entries detected by the chosen relative pairing.

Chapter 1

The Upper Half-Plane and the Modular Group

1.1 Möbius transformations and the role of the center

This section is in preparation.

1.2 Hyperbolic metric, measure, distance, and geodesics

This section is in preparation.

1.3 Reduction to the standard fundamental domain

This section is in preparation.

1.4 The cusp, elliptic points, and stabilizers

This section is in preparation.

1.5 Quotient integration and the area calculation

This section is in preparation.

Chapter 2

Fourier Analysis, Mellin Transforms, and Unfolding

2.1 Fourier series and coefficient extraction

This section is in preparation.

2.2 Fourier transforms and Poisson summation

This section is in preparation.

2.3 Gamma integrals and Mellin inversion

This section is in preparation.

2.4 A general unfolding lemma

This section is in preparation.

2.5 Convergence, domination, and truncation

This section is in preparation.

Part II

Holomorphic forms and trace formulas

Chapter 3

Holomorphic Modular Forms

3.1 Automorphy factors and slash actions

This section is in preparation.

3.2 The q-variable and holomorphy at infinity

This section is in preparation.

3.3 Modular forms, cusp forms, and Fourier expansions

This section is in preparation.

3.4 Growth and cusp decay

This section is in preparation.

3.5 The Petersson inner product

This section is in preparation.

Chapter 4

Explicit Forms, Zeros, and Dimensions

4.1 Holomorphic Eisenstein series and their coefficients

This section is in preparation.

4.2 E_4, E_6, and the discriminant

This section is in preparation.

4.3 The valence formula, including elliptic points

This section is in preparation.

4.4 The ring of level-one modular forms

This section is in preparation.

4.5 Dimension formulas and explicit small-weight bases

This section is in preparation.

Chapter 5

Hecke Operators and Their Two Normalizations

5.1 Integral matrices, determinant correspondences, and coset representatives

This section is in preparation.

5.2 The classical operators and their Fourier action

This section is in preparation.

5.3 Hecke multiplication and prime-power recurrences

This section is in preparation.

5.4 Adjointness and simultaneous eigenbases

This section is in preparation.

5.5 Classical versus normalized operators; eigenvalue conventions

This section is in preparation.

Chapter 6

Poincaré Series, Kloosterman Sums, and Bessel Functions

6.1 Automorphization of a seed and Poincaré series

This section is in preparation.

6.2 Absolute and local uniform convergence

This section is in preparation.

6.3 Fourier coefficients as Petersson pairings

This section is in preparation.

6.4 Coset decomposition and Kloosterman sums

This section is in preparation.

6.5 The Bessel integral and the full Fourier expansion

This section is in preparation.

Chapter 7

The Petersson Trace Formula

7.1 The definitive orthonormal-basis statement

This section is in preparation.

7.2 The Poincaré-series proof

This section is in preparation.

7.3 Hecke-normalized and harmonic-weight versions

This section is in preparation.

7.4 Convergence, small weights, and vanishing spaces

This section is in preparation.

7.5 Orthogonality, second moments, and consistency checks

This section is in preparation.

Chapter 8

Holomorphic Kernels and Linear Functionals

8.1 Finite-dimensional reproducing kernels

This section is in preparation.

8.2 The holomorphic kernel on the upper half-plane

This section is in preparation.

8.3 Automorphization and the cuspidal kernel

This section is in preparation.

8.4 Kernels of Hecke operators

This section is in preparation.

8.5 Linear functionals, Fourier extraction, and relative pairings

This section is in preparation.

Chapter 9

Quadratic Forms, Conjugacy Classes, and Class Numbers

9.1 Binary quadratic forms and discriminants

This section is in preparation.

9.2 Reduction and equivalence

This section is in preparation.

9.3 Automorphisms and weighted class numbers

This section is in preparation.

9.4 Matrices of determinant n and discriminant t^2-4n

This section is in preparation.

9.5 Scalar, elliptic, parabolic, and hyperbolic cases

This section is in preparation.

Chapter 10

The Eichler–Selberg Trace Formula

10.1 What the Hecke trace measures

This section is in preparation.

10.2 The holomorphic-kernel calculation

This section is in preparation.

10.3 Geometric contributions and correction terms

This section is in preparation.

10.4 Dimension and one-dimensional-space checks

This section is in preparation.

10.5 A complementary period-polynomial proof

This section is in preparation.

Part III

Maass forms and spectral theory

Chapter 11

The Hyperbolic Laplacian

11.1 Elementary spectral models

This section is in preparation.

11.2 The hyperbolic Laplacian and invariance

This section is in preparation.

11.3 Hilbert spaces and unbounded operators

This section is in preparation.

11.4 Green identities, domains, and self-adjointness

This section is in preparation.

11.5 Regularity, compactness, and the cusp

This section is in preparation.

Chapter 12

Maass Forms and Their Fourier Expansions

12.1 Automorphic eigenfunctions and cusp conditions

This section is in preparation.

12.2 Separation of variables

This section is in preparation.

12.3 The K-Bessel Fourier expansion

This section is in preparation.

12.4 Hecke operators and Fourier–Hecke relations

This section is in preparation.

12.5 Parity, spectral parameters, and normalizations

This section is in preparation.

Chapter 13

Eisenstein Series

13.1 Construction and initial convergence

This section is in preparation.

13.2 Fourier expansion and the constant term

This section is in preparation.

13.3 The scattering coefficient and zeta factors

This section is in preparation.

13.4 Continuation, functional equation, and residues

This section is in preparation.

13.5 Incomplete Eisenstein series

This section is in preparation.

Chapter 14

Truncation and the Maass–Selberg Relations

14.1 Two forms of truncation

This section is in preparation.

14.2 Boundary terms from Green identities

This section is in preparation.

14.3 The Maass–Selberg calculation

This section is in preparation.

14.4 Coincident parameters and limiting cases

This section is in preparation.

14.5 Scattering identities and spectral measure

This section is in preparation.

Chapter 15

The Spectral Decomposition of the Modular Surface

15.1 The discrete spectrum and the constant eigenfunction

This section is in preparation.

15.2 The cuspidal subspace

This section is in preparation.

15.3 The Eisenstein contribution

This section is in preparation.

15.4 Completeness and spectral expansion

This section is in preparation.

15.5 Parseval, Plancherel, and modes of convergence

This section is in preparation.

Chapter 16

L-Functions, Rankin–Selberg Integrals, and Spectral Weights

16.1 Normalized L-functions and Euler/Hecke conventions

This section is in preparation.

16.2 Holomorphic Mellin integrals and functional equations

This section is in preparation.

16.3 Maass Mellin integrals and parity

This section is in preparation.

16.4 Rankin–Selberg unfolding and residues

This section is in preparation.

16.5 Norm identities and harmonic spectral weights

This section is in preparation.

Part IV

Kernels and Selberg

Chapter 17

Automorphic Kernels and the Pre-Trace Formula

17.1 Point-pair invariants and radial kernels

This section is in preparation.

17.2 The Selberg transform and its inverse

This section is in preparation.

17.3 Periodization and convergence

This section is in preparation.

17.4 Action on discrete and Eisenstein eigenfunctions

This section is in preparation.

17.5 The pre-trace identity

This section is in preparation.

Chapter 18

Conjugacy Classes, Closed Geodesics, and Orbital Integrals

18.1 Conjugacy classes and centralizers

This section is in preparation.

18.2 Orbital unfolding

This section is in preparation.

18.3 Identity and elliptic terms

This section is in preparation.

18.4 Primitive closed geodesics and hyperbolic integrals

This section is in preparation.

18.5 Parabolic classes and cusp geometry

This section is in preparation.

Chapter 19

The Selberg Trace Formula

19.1 Why the naive diagonal trace diverges

This section is in preparation.

19.2 Simultaneous geometric and spectral truncation

This section is in preparation.

19.3 Cancellation and the limiting argument

This section is in preparation.

19.4 The complete modular-group formula

This section is in preparation.

19.5 Test functions and equivalent versions

This section is in preparation.

Chapter 20

Applications of Selberg's Formula

20.1 Designing spectral test functions

This section is in preparation.

20.2 Weyl law and scattering contributions

This section is in preparation.

20.3 The Selberg zeta function

This section is in preparation.

20.4 A first prime geodesic theorem

This section is in preparation.

20.5 Smoothing, unsmoothing, and stronger estimates

This section is in preparation.

Part V

Fourier periods and Kuznetsov

Chapter 21

Fourier Periods and the Pre-Kuznetsov Formula

21.1 Fourier coefficients as periods

This section is in preparation.

21.2 Poincaré series with controlled seeds

This section is in preparation.

21.3 Unfolded spectral pairings

This section is in preparation.

21.4 Unipotent double cosets and Kloosterman sums

This section is in preparation.

21.5 The preliminary relative trace identities

This section is in preparation.

Chapter 22

The Bessel Transforms of Kuznetsov

22.1 Same-sign kernels

This section is in preparation.

22.2 Opposite-sign kernels

This section is in preparation.

22.3 Transform inversion and admissibility

This section is in preparation.

22.4 Contour shifts, residues, and holomorphic terms

This section is in preparation.

22.5 Decay and differentiation estimates

This section is in preparation.

Chapter 23

The Kuznetsov Trace Formulas

23.1 The spectral-to-arithmetic statement

This section is in preparation.

23.2 The arithmetic-to-spectral statement

This section is in preparation.

23.3 Signs, parity, continuous and holomorphic terms

This section is in preparation.

23.4 Fourier and Hecke normalization dictionaries

This section is in preparation.

23.5 Orthogonality, spectral moments, and Kloosterman averages

This section is in preparation.

Part VI

Classical relative trace formulas

Chapter 24

Periods, Double Cosets, and Closed-Geodesic Trace Formulas

24.1 Period functionals and smoothing kernels

This section is in preparation.

24.2 Relative unfolding and double cosets

This section is in preparation.

24.3 Fourier periods revisited

This section is in preparation.

24.4 A closed-geodesic relative trace formula

This section is in preparation.

24.5 Stabilizers, orbital integrals, and the continuous spectrum

This section is in preparation.

Chapter 25

Relative Trace Formulas for Central L-Values

25.1 Mellin periods and functional-equation signs

This section is in preparation.

25.2 Double periods of the holomorphic kernel

This section is in preparation.

25.3 The geometric decomposition and regularization

This section is in preparation.

25.4 Hecke insertions and weighted moments

This section is in preparation.

25.5 An explicit formula and a first application

This section is in preparation.

Part VII

Weighted trace formulas

Chapter 26

Weighted Diagonal Integrals and Spectral Matrix Elements

26.1 Compactly supported weights

This section is in preparation.

26.2 Spectral expansion of the weighted diagonal

This section is in preparation.

26.3 Multiplication operators and trace-class hypotheses

This section is in preparation.

26.4 Weighted traces versus period pairings

This section is in preparation.

26.5 First localized spectral averages

This section is in preparation.

Chapter 27

Eigenfunction Weights and Geodesic Periods

27.1 Cuspidal Laplace eigenfunctions as weights

This section is in preparation.

27.2 Weighted orbital integrals

This section is in preparation.

27.3 Averaging the eigenfunction equation

This section is in preparation.

27.4 Geodesic periods and eigenvalue-dependent transforms

This section is in preparation.

27.5 The full cuspidal-weight formula

This section is in preparation.

Chapter 28

Regularization and the Zelditch–Biró Formulas

28.1 Noncuspidal weights and their constant terms

This section is in preparation.

28.2 Weighted truncation and regularized pairings

This section is in preparation.

28.3 Poles, exceptional parameters, and boundary contributions

This section is in preparation.

28.4 A normalization and transform dictionary

This section is in preparation.

28.5 The exact common scope of the comparison

This section is in preparation.

Part VIII

Connections and synthesis

Chapter 29

Classical Weight-k Operators and Hecke-Twisted Trace Formulas

29.1 Unitary weight-k automorphy

This section is in preparation.

29.2 Weight-k Laplacians

This section is in preparation.

29.3 Raising and lowering operators

This section is in preparation.

29.4 The holomorphic contribution in the correct weighted space

This section is in preparation.

29.5 Hecke-twisted kernels and trace comparisons

This section is in preparation.

Chapter 30

A Dictionary Between Trace Formulas

30.1 A common inventory of kernels and functionals

This section is in preparation.

30.2 Direct specializations

This section is in preparation.

30.3 Common-kernel derivations

This section is in preparation.

30.4 Comparisons requiring weights, projections, or Hecke operators

This section is in preparation.

30.5 Information retained and discarded; the bridge to Volume 1b

This section is in preparation.

Appendix A

Analytic Foundations

This appendix is in preparation.

Appendix B

Special Functions

This appendix is in preparation.

Appendix C

Functional Analysis and Differential Operators

This appendix is in preparation.

Appendix D

Integral Transforms

This appendix is in preparation.

Appendix E

Arithmetic Estimates and Summation Tools

This appendix is in preparation.

Appendix F

Period Polynomials

This appendix is in preparation.

Appendix G

Normalization Dictionary

This appendix is in preparation.

Appendix H

Supplementary Proofs

This appendix is in preparation.

Appendix I

Notational Index

This index collects the notation used in Chapter 0, grouped by mathematical context. The links in the final column lead to the relevant definition or convention. Local meanings are recorded separately when the same letter serves different purposes.

Matrices and traces

Symbol	Meaning and convention	See
$A = (a_{ij})$	Complex matrix, with entry a_{ij} in row i, column j.	0.1
A^{T}	Transpose: $(A^{\mathsf{T}})_{ij} = a_{ji}$; entries are not conjugated. A transposed row vector is a column vector.	0.1
I_n	Identity matrix of size n; its trace is n.	0.1
$M_n(\mathbb{C})$	Complex $n \times n$ matrices.	0.1
$\det(tI_n - A)$	Characteristic polynomial of A.	0.1
λ_j	Eigenvalues, repeated according to algebraic multiplicity.	0.1
$\operatorname{tr}(A)$	Sum of the diagonal entries; the empty sum is zero.	0.1
$\operatorname{tr}(T)$	Trace of a finite-dimensional endomorphism, computed in any basis.	0.1

Function spaces and finite kernels

Symbol	Meaning and convention	See
(X, μ)	Sigma-finite measure space in the kernel model.	0.2
H, $L^2(X, \mu; \mathbb{C})$	Complex square-integrable functions, identified when equal almost everywhere.	0.2
$\langle f, g \rangle_{L^2(X,\mu)}$	Inner product $\int_X f\overline{g}\, d\mu$ on the complex space H, linear in the first argument.	0.2
$\|f\|_{L^2(X,\mu)}$	Norm $\langle f, f \rangle^{1/2}_{L^2(X,\mu)}$ on H; both space and measure appear in the subscript.	0.2
$u_1, \ldots, u_N$	Fixed orthonormal family with chosen measurable representatives; $N = 0$ is allowed.	0.2

V, $V^{\perp}$	Span of the family and its orthogonal complement in H.	0.2
T, λ_j	Operator with $Tu_j = \lambda_j u_j$ and $T\|_{V^{\perp}} = 0$ in the eigenbasis model.	0.2
$K_T(x, y)$	Spectral kernel $\sum_j \lambda_j u_j(x)\overline{u_j(y)}$ from the chosen orthonormal eigenfunctions.	0.2
ν_j	Eigenvalues of the general matrix A, counted with algebraic multiplicity; the u_j need not be eigenfunctions.	0.2
$K_A(x, y)$	Finite kernel $\sum_{i,j} a_{ij} u_i(x)\overline{u_j(y)}$; the output variable comes first.	0.2
T_A	Operator with kernel K_A; it acts as A on V and vanishes on $V^{\perp}$.	0.2
$\mathrm{tr}(T_A)$	Trace of the restriction $T_A\|_V$; likewise $\mathrm{tr}(T) = \mathrm{tr}(T\|_V)$ in the eigenbasis model.	0.2
$K_A(x, x)$	Diagonal supplied by the finite kernel formula, independent μ-almost everywhere of the u_j representatives. Arbitrary $(\mu \otimes \mu)$-null changes to a kernel need not preserve its diagonal integral.	0.2

Ordinary traces and relative pairings

Symbol	Meaning and convention	See
$\mathcal{T}(K_A)$	Ordinary diagonal integral $\int_X K_A(x, x)\, d\mu(x) = \mathrm{tr}(A)$.	0.3
$P_p(u)$	Test integral $\langle u, p\rangle_{L^2(X,\mu)}$, linear in $u \in H$ for fixed $p \in H$; an elementary model of a period.	0.3
$\mathcal{R}_{p,q}(K_A)$	Relative pairing $\langle T_A q, p\rangle_{L^2(X,\mu)}$, with input $q \in H$ and output tested against $p \in H$.	0.3
$1_{\{r\}}$	Indicator of the point r in the two-point counting-measure example; equals 1 at r and 0 at the other point.	0.3

Weights and multiplication operators

Symbol	Meaning and convention	See
$L^{\infty}(X, \mu; \mathbb{C})$	Essentially bounded complex functions, identified when equal almost everywhere.	0.3
w, M_w	A weight $w \in L^{\infty}$ and its multiplication operator $M_w f = wf$.	0.3
P	Orthogonal projection onto V; distinct from the scalar period P_p.	0.3
S_w	Endomorphism $PM_w T_A\|_V$, obtained by projecting the weighted output back to V.	0.3
$\mathcal{W}_w(K_A)$	Weighted diagonal integral $\int_X w(x) K_A(x, x)\, d\mu(x) = \mathrm{tr}(S_w)$.	0.3